Introduction to Integrable Models

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Chapter 1

Introduction and Motivation

Integrable models are exactly solvable models of many-body systems inspired from statistical mechanics or solid state physics. They are usually the result of some simplifications and approximations of a real-world physical system. As these models describe things made up of atoms, they are inherently discrete (lattice models). With some exceptions, most known integrable models are however one-dimensional, which means that we have quite some way to go before we can solve realistic systems. The models we can solve fully are a bit like very simple table-top experiments. By studying these systems, we hope to gain valuable tools for tackling more complicated systems.

The concept of integrability shows up in the context of classical mechanics, classical field theory (*e.g.* classical sine–Gordon model), but also in conjunction with combinatorial objects, such as for example triangulations, trees, tilings, alternating sign matrices, and plane partitions. Integrable models are also relevant for the high energy physicist, as they are intimately connected to supersymmetric quantum field theories and string theory.

In these lectures, we will focus largely on one class of quantum integrable models, namely *spin chains*.

In 1931, Hans Bethe solved the $\text{XXX}_{1/2}$ or Heisenberg spin chain. His ansatz can be generalized to many more systems and is the basis of the field of integrable systems. Colloquially, people often equate "Bethe-solvable" with "integrable", but the set of integrable models is marginally bigger.

The example with which we will start our journey is the one of a *1d ferromagnet*. Here, one studies a linear chain of L identical atoms with only next-neighbor interactions. Each atom has one electron in an outer shell (all other shells being complete). These electrons can either be in the state of spin up ($\uparrow$) or down ($\downarrow$). At first order, the Coulomb- and magnetic interactions result in the *exchange interaction* in which the states of neighboring spins are interchanged:

$$\uparrow\downarrow \quad \leftrightarrow \quad \downarrow\uparrow \tag{1.1}$$

In a given spin configuration of a spin chain, interactions can happen at all the anti-parallel pairs. Take for example the configuration

$$\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow \; . \tag{1.2}$$

It contains five anti-parallel pairs on which the exchange interaction can act, giving rise to five new configurations.

For simplicity, we will be studying the periodic chain. Bethe posed himself the question of finding the *spectrum* and *energy eigenfunctions* of this spin chain. We will study his ansatz in detail in the next chapter. His method is a little gem and studying it is bound to lift the morale of any theoretical physicist!
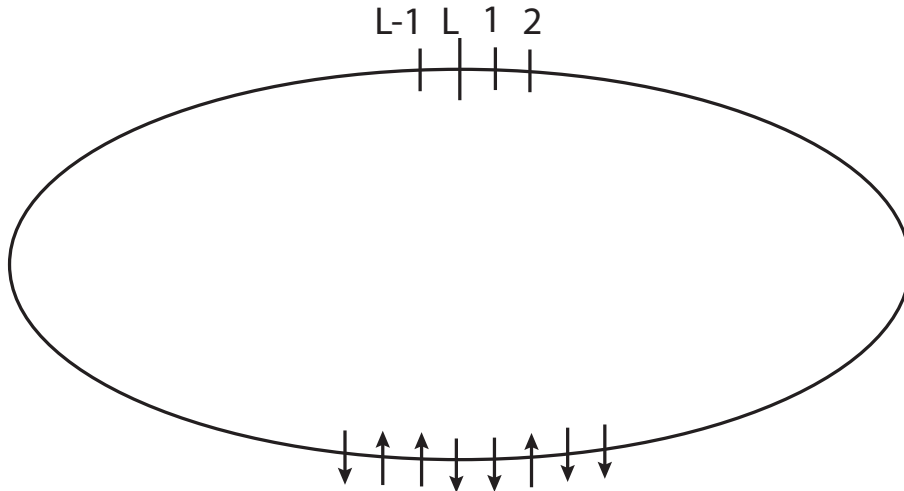
L-1 L   1   2

Figure 1.1: Periodic spin chain

There are many generalizations to this simplest of all spin chains which can still be solved by versions of Bethe's ansatz:

- different *boundary conditions*: periodic, anti-periodic, open, kink, ...

- *anisotropic* models: XXZ chain, where the *z*-direction is singled out by a magnetic field in this direction; XYZ model.

- different choice of *symmetry algebra*. The spin 1/2 spin chain corresponds to $su(2)$, but any Lie algebra or even super-algebra can be chosen instead.

- for the *rank* of the symmetry algebra $r > 1$, there are more particle species on the chain, *e.g.* ↑, ↓, ∘ (hole) of the *tJ*–model, where ↑, ↓ are fermionic while ∘ is bosonic.

- each site of the spin chain can carry a different *representation* of the symmetry algebra.

- on each site another parameter, the so-called *inhomogeneity* can be turned on.

We still start this lecture series with Bethe's original treatment of the $XXX_{1/2}$ spin chain, the so-called *coordinate* Bethe ansatz. It has the virtue of being very intuitive and will give us a good understanding of the physics of the spin chain. Its drawback is that it cannot be generalized very much. We will therefore graduate to the more abstract, but much more powerful *algebraic* Bethe ansatz after we have finished with the simple case at hand. Here, we will rederive the $XXX_{1/2}$ case by algebraic means and then generalize it to general spin s. I will also briefly sketch further generalizations, such as spin chains with higher rank symmetry algebra. Towards the end of this lecture series, I will cover some of the relations between integrable models and supersymmetric gauge theories which are relevant for particle physics, in particular the gauge/Bethe correspondence.

The field of integrable models is very rich and there are many more interesting models and complementary ways of studying them that would deserve our attention, but in this short course, we have to content ourselves with the topics outlined above.

Chapter 2

The coordinate Bethe ansatz

As discussed in the introduction, we want to find the energy eigenfunctions and eigenvalues of a 1d magnet. Consider a closed chain of identical atoms with each one external electron which can be in the state of spin up or down and only next-neighbor interactions — the XXX$_{1/2}$ spin chain. Its Hamiltonian (Heisenberg 1926) is given by

$$H = -J \sum_{n=1}^{L} \Pi_{n,n+1}, \tag{2.1}$$

where J is the exchange integral[1] and $\Pi_{n,n+1}$ is the permutation operator of states at positions n, $n+1$. Let us write down the spin operator at position n on the spin chain:

$$\vec{S}_n = (S_n^x, S_n^y, S_n^z) = \tfrac{1}{2} \vec{\sigma}_n, \tag{2.2}$$

where σ_n are the Pauli matrices for spin $1/2$:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

In terms of the spin operators, the permutation operator is given by

$$\Pi_{n,n+1} = -\tfrac{1}{2}(1 + \vec{\sigma}_n \vec{\sigma}_{n+1}). \tag{2.4}$$

In terms of the spin operators, the Hamiltonian (2.1) becomes

$$\begin{aligned} H &= -J \sum_{n=1}^{L} \vec{S}_n \vec{S}_{n+1} \\ &= -J \sum_{n=1}^{L} \tfrac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + S_n^z S_{n+1}^z, \end{aligned} \tag{2.5}$$

where $S^{\pm} = S_n^x \pm i S_n^y$ are the *spin flip operators*. The term in parentheses corresponds to the *exchange interaction* which exchanges neighboring spin states.

The spin flip operators act as follows on the spins:

$$\begin{aligned} S_k^+ |\ldots \uparrow \ldots\rangle &= 0, & S_k^+ |\ldots \downarrow \ldots\rangle &= |\ldots \uparrow \ldots\rangle, \\ S_k^- |\ldots \uparrow \ldots\rangle &= |\ldots \downarrow \ldots\rangle, & S_k^- |\ldots \downarrow \ldots\rangle &= 0, \\ S_k^z |\ldots \uparrow \ldots\rangle &= \tfrac{1}{2} |\ldots \uparrow \ldots\rangle, & S_k^z |\ldots \downarrow \ldots\rangle &= -\tfrac{1}{2} |\ldots \downarrow \ldots\rangle. \end{aligned} \tag{2.6}$$

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[1] $-J > 0$: ferromagnet, spins tend to align, $-J < 0$: anti-ferromagnet, spins tend to be anti-parallel.

The spin operators have the following commutation relations:

$$[S_n^z, S_{n'}^\pm] = \pm S_n^\pm \delta_{nn'}, \quad [S_n^+, S_{n'}^-] = 2S_n^z \delta_{nn'}. \tag{2.7}$$

For the *closed* chain, the sites n and $n+L$ are identified:

$$\vec{S}_{L+1} = \vec{S}_1. \tag{2.8}$$

So far, we have discussed the *isotropic* spin chain. In the anisotropic case, a magnetic field is turned on in the z-direction:

$$H_\Delta = -J \sum_{n=1}^L S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta(S_n^z S_{n+1}^z - \tfrac{1}{4}). \tag{2.9}$$

This is the XXZ spin chain. Δ is the anisotropy parameter, where $\Delta = 1$ is the isotropic case. The most general model in this respect is the XYZ spin chain:

$$H_{\Delta,\Gamma} = -\sum_{n=1}^L J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z. \tag{2.10}$$

It has two anisotropy parameters Δ, Γ which fulfill the ratios

$$J_x : J_y : J_z = 1 - \Gamma : 1 + \Gamma : \Delta. \tag{2.11}$$

In the following, we will however concentrate on the isotropic case.

Let us define the *ferromagnetic reference state*

$$| \uparrow\uparrow \dots \uparrow\uparrow \rangle = |\Omega\rangle. \tag{2.12}$$

H acts on a Hilbert space of dimension 2^L, given that each site on the chain can be in one of two states, which is spanned by the orthogonal basis vectors

$$|\Omega(n_1, \dots, n_N)\rangle = S_{n_1}^- \dots S_{n_N}^- |\Omega\rangle, \tag{2.13}$$

which are vectors with N down spins ($0 \leq N \leq L$) in the positions $n_1, \dots, n_N$, where we always take $1 \leq n_1 < n_2 < \dots < n_N \leq L$.

In order to diagonalize the Heisenberg model, two symmetries will be of essential importance:

- the conservation of the z–component of the total spin,

$$[H, S^z] = 0, \quad S^z = \sum_{n=1}^L S_n^z. \tag{2.14}$$

  This remains also true for the XXZ spin chain Hamiltonian H_Δ.

- the translational symmetry, *i.e.* the invariance of H with respect to discrete translations by any number of lattice spacings. This symmetry results from the periodic boundary conditions we have imposed.

As the exchange interaction only moves down spins around, the number of down spins in a basis vector is not changed by the action of H. Acting with H on $|\Omega(n_1, \dots, n_N)\rangle$ thus yields a linear combination of basis vectors with N down spins. It is therefore possible to *block-diagonalize* H by sorting the basis vectors by the quantum number $S^z = L/2 - N$.

Let us start by considering the subsector with $N = 0$. It contains only one single basis vector, namely $|\Omega\rangle$, which is an eigenvector of H as there are no antiparallel spins for the exchange interaction to act on:

$$H|\Omega\rangle = E_0|\Omega\rangle, \quad E_0 = -J\tfrac{L}{4}. \tag{2.15}$$

Next we consider the sector with $N = 1$. As the down spin can be in each of the lattice sites, this subspace is spanned by

$$|\Omega(n)\rangle = S_n^-|\Omega\rangle. \tag{2.16}$$

In order to diagonalize this block, we must invoke the translational symmetry. We can construct translationally invariant basis vectors as follows:

$$|\psi\rangle = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn}|\Omega(n)\rangle, \quad k = \frac{2\pi m}{L}, \, m = 0, 1, \ldots, L - 1. \tag{2.17}$$

The $|\psi\rangle$ with wave number k are eigenvectors of the translation operator with eigenvalue e^{ik} and eigenvectors of H with eigenvalues

$$E = -J(\tfrac{L}{4} - 1 - \cos k), \tag{2.18}$$

or in terms of E_0,

$$E - E_0 = J(1 - \cos k). \tag{2.19}$$

The $|\psi\rangle$ are so-called *magnon* excitations: the ferromagnetic ground state is periodically perturbed by a *spin wave* with wave length $2\pi/k$.

So far, we have block-diagonalized H and diagonalized the sectors $N = 0, 1$ by symmetry considerations alone. The invariant subspaces with $2 \leq N \leq L/2$ however are not completely diagonalized by the translationally invariant basis.

In order to remedy this situation, we will now study Bethe's ansatz, again for the case $N = 1$.

Bethe ansatz for the one-magnon sector. We can write the eigenvectors of H in this sector as

$$|\psi\rangle = \sum_{n=1}^{L} a(n)|\Omega(n)\rangle. \tag{2.20}$$

Plugging this into the eigenvalue equation results in a set of conditions for $a(n)$:

$$2\left[E + \frac{JL}{4}\right]a(n) = J\left[2a(n) - a(n-1) - a(n+1)\right], \quad n = 1, 2, \ldots, L. \tag{2.21}$$

On top of this, we have the periodic boundary conditions

$$a(n + L) = a(n). \tag{2.22}$$

The L linearly independent solutions to the difference equation Eq. (2.21) are given by

$$a(n) = e^{ikn}, \quad k = \frac{2\pi}{L}m, \, m = 0, 1, \ldots, L - 1. \tag{2.23}$$

Little surprisingly, these are the same solutions we had found before. But now we can apply the same procedure to the case $N = 2$.

Bethe ansatz for the two-magnon sector. This invariant subspace has dimension $L(L - 1)/2$. We want to determine $a(n_1, n_2)$ for the eigenstates of the form

$$|\psi\rangle = \sum_{1 \leq n_1 < n_2 \leq L} a(n_1, n_2)|\Omega(n_1, n_2)\rangle. \tag{2.24}$$

Bethe's preliminary ansatz is given by

$$a(n_1, n_2) = A\, e^{i(k_1 n_1 + k_2 n_2)} + A'\, e^{i(k_1 n_2 + k_2 n_1)}. \tag{2.25}$$

The first term is called the *direct term* and represents an incoming wave, while the second term is called the *exchange term* and represents an outgoing wave. Indeed, the expression looks like the superposition of two magnons, however, the flipped spins must always be in different lattice sites. Asymptotically, we can only have the direct and exchange terms for two magnons. Bethe's ansatz postulates that this asymptotic form remains true in general.

Let us now plug the eigenstates $|\psi\rangle$ of the form given in Eq. (2.25) into the eigenvalue equation. There are two cases to consider separately, namely the two down spins not being adjacent, and the two down spins being adjacent:

$$2(E - E_0)\, a(n_1, n_2) = J\,[4\, a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1 + 1, n_2) \tag{2.26}$$
$$-a(n_1, n_2 - 1) - a(n_1, n_2 + 1)], \quad n_2 > n_1 + 1,$$
$$2(E - E_0)\, a(n_1, n_2) = J\,[2a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1, n_2 + 1)], \quad n_2 = n_1 + 1. \tag{2.27}$$

Equations (2.26) are satisfied by $a(n_1, n_2)$ of the form Eq. (2.25) with arbitrary A, A', k_1, k_2 for both $n_2 > n_1 + 1$ and $n_2 = n_1 + 1$ if the energies fulfill

$$E - E_0 = J \sum_{j=1,2} (1 - \cos k_j). \tag{2.28}$$

Equation (2.27) on the other hand is not automatically satisfied. Subtracting equation (2.27) from equation (2.26) for the case $n_2 = n_1 + 1$ leads to L conditions, known as the *meeting conditions*:

$$2\, a(n_1, n_1 + 1) = a(n_1, n_1) + a(n_1 + 1, n_1 + 1). \tag{2.29}$$

Clearly, the expressions $a(n_1, n_1)$ have no physical meaning, as the two down spins cannot be at the same site, but are defined formally by the ansatz Eq. (2.25). Thus the $a(n_1, n_2)$ solve Eq. (2.26), (2.27) if they have the form Eq. (2.25) and fulfill Eq. (2.29). Plugging Eq. (2.25) into Eq. (2.29) and taking the ratio, we arrive at

$$\frac{A}{A'} =: e^{i\theta} = -\frac{e^{i(k_1 + k_2)} + 1 - 2\, e^{ik_1}}{e^{i(k_1 + k_2)} + 1 - 2\, e^{ik_2}}. \tag{2.30}$$

We see thus that as a result of the magnon interaction, we get an extra phase factor in the Bethe ansatz Eq. (2.25):

$$a(n_1, n_2) = e^{i(k_1 n_1 + k_2 n_2 + \frac{1}{2}\theta_{12})} + e^{i(k_1 n_2 + k_2 n_1 + \frac{1}{2}\theta_{21})}, \tag{2.31}$$

where $\theta_{12} = -\theta_{21} = \theta$, or written in the real form,

$$2 \cot \theta/2 = \cot k_1/2 - \cot k_2/2. \tag{2.32}$$

k_1, k_2 are the momenta of the Bethe ansatz wave function. The translational invariance of $|\psi\rangle$,

$$a(n_1, n_2) = a(n_2, n_1 + L) \tag{2.33}$$

is satisfied if

$$e^{ik_1 L} = e^{i\theta}, \quad e^{ik_2 L} = e^{-i\theta}, \tag{2.34}$$

which, after taking the logarithm, is equivalent to

$$L k_1 = 2\pi \tilde{\lambda}_1 + \theta, \quad L k_2 = 2\pi \tilde{\lambda}_2 + \theta, \tag{2.35}$$

where $\tilde{\lambda}_i \in \{0, 1, \dots, L-1\}$ are the *Bethe quantum numbers* which fulfill

$$k = k_1 + k_2 = \tfrac{2\pi}{L}(\tilde{\lambda}_1 + \tilde{\lambda}_2). \tag{2.36}$$

We have seen that the expression for the energies, Eq. (2.28), is reminiscent of two superimposed magnons. The magnon interaction is reflected in the phase shift θ and the deviation of the momenta k_1, k_2 from the one-magnon wave numbers. We will see that the magnons either *scatter* off each other or form *bound states*. In the following lectures, we will be mostly interested in the form of the Bethe equations themselves, and not so much in their explicit solutions. But before treating the general N magnon case, we will nonetheless quickly review the properties of the Bethe eigenstates for $N = 2$.

We need to identify all pairs $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ that satisfy the Bethe Equations (2.32) and (2.35). Allowed pairs are restricted to $0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq L-1$. Switching $\tilde{\lambda}_1$ with $\tilde{\lambda}_2$ interchanges k_1 and k_2 and leads to the same solution. $L(L+1)/2$ pairs meet the restriction, however only $L(L-1)/2$ of them produce solutions, which corresponds to the size of the Hilbert space. There are three distinct classes of solutions:

1. *One of the Bethe quantum numbers is zero:* $\tilde{\lambda}_1 = 0$, $\tilde{\lambda}_2 = 0, 1, \dots, L-1$. There exists a *real* solution for all L combinations $k_1 = 0$, $k_2 = 2\pi\tilde{\lambda}_2/L$, $\theta = 0$. These solutions have the same dispersion relation as the one-magnon states in the subspace $N = 1$.

2. $\tilde{\lambda}_1, \tilde{\lambda}_2 \neq 0$, $\tilde{\lambda}_2 - \tilde{\lambda}_1 \geq 2$. There are $L(L-5)/2 + 3$ such pairs and each gives a solution with real k_1, k_2. These solutions represent nearly free superpositions of two one-magnon states.

3. $\tilde{\lambda}_1, \tilde{\lambda}_2 \neq 0$, $\tilde{\lambda}_1, \tilde{\lambda}_2$ *are either equal or differing by unity.* There are $2L - 3$ such pairs, but only $L - 3$ yield solutions. Most are complex, $k_1 := k/2 + iv$, $k_2 := k/2 - iv$, $\theta := \phi + i\chi$. These solutions correspond to two-magnon bound states. They exhibit an enhanced probability that the two flipped spins are on neighboring sites.

The number of solutions adds up to the dimension of the Hilbert space. The first and second class of solutions correspond to two-magnon scattering states.

Bethe ansatz for the N–magnon sector. We are finally ready to tackle the general case with an unrestricted number $N \leq L$ of down spins. This subspace has dimension $L!/((L-N)! N!)$. The eigenstates have the form

$$|\psi\rangle = \sum_{1 \leq n_1 < \dots < n_N \leq L} a(n_1, \dots, n_N) |\Omega(n_1, \dots, n_N)\rangle. \tag{2.37}$$

Here, we have N momenta k_j and one phase angle $\theta_{ij} = -\theta_{ji}$ for each pair (k_i, k_j). The *Bethe ansatz* now has the form

$$a(n_1, \dots, n_N) = \sum_{\mathcal{P} \in S_N} \exp\left(i \sum_{j=1}^{N} k_{p(j)} n_j + \tfrac{i}{2} \sum_{i<j} \theta_{p(i)p(j)} \right), \tag{2.38}$$

where $\mathcal{P} \in S_N$ are the $N!$ permutations of $\{1, 2, \ldots, N\}$. From the eigenvalue equation, we again get the two kinds of difference equations (the first for no adjacent down spins, the second for one pair of adjacent down spins),

$$2[E - E_0] \, a(n_1, \ldots, n_N) = J \sum_{i=1}^{N} \sum_{\sigma = \pm 1} [a(n_1, \ldots, n_N) - a(n_1, \ldots, n_{i+\sigma}, \ldots, n_N)], \qquad (2.39)$$

$$\text{if } n_{j+1} > n_j + 1, \ j = 1, \ldots, N,$$

$$2[E - E_0] \, a(n_1, \ldots, n_N) = J \sum_{i \neq j_\alpha, j_\alpha + 1} \sum_{\sigma = \pm 1} [a(n_1, \ldots, n_N) - a(n_1, \ldots, n_{i+\sigma}, \ldots, n_N)] \qquad (2.40)$$

$$+ J \sum_\alpha [2 \, a(n_1, \ldots, n_N) - a(n_1, \ldots, n_{j_\alpha} - 1, n_{j_\alpha + 1}, \ldots, n_N)$$

$$- a(n_1, \ldots, n_{j_\alpha}, n_{j_\alpha + 1} + 1, \ldots, n_N)],$$

$$\text{if } n_{j_\alpha} + 1 = n_{j_\alpha + 1}, \ n_{j+1} > n_j + 1, \ j \neq j_\alpha.$$

The coefficients $a(n_1, \ldots, n_N)$ are solutions of Equations (2.39), (2.40) for the energy

$$E - E_0 = J \sum_{j=1}^{N} (1 - \cos k_j) \qquad (2.41)$$

if they have the form Eq. (2.38) and fulfill the N meeting conditions

$$2 \, a(n_1, \ldots, n_{j_\alpha}, n_{j_\alpha} + 1, \ldots, n_N) = a(n_1, \ldots, n_{j_\alpha}, n_{j_\alpha}, \ldots, n_N)$$

$$+ a(n_1, \ldots, n_{j_\alpha} + 1, n_{j_\alpha} + 1, \ldots, n_N), \qquad (2.42)$$

for $\alpha = 1, \ldots, N$. This relates the phase angles to the (not yet determined) k_j:

$$e^{i\theta_{ij}} = -\frac{e^{i(k_i + k_j)} + 1 - 2 \, e^{ik_i}}{e^{i(k_i + k_j)} + 1 - 2 \, e^{ik_j}}, \qquad (2.43)$$

or, in the real form

$$2 \cot \theta_{ij}/2 = \cot k_i/2 - \cot k_j/2, \quad i, j = 1, \ldots, N. \qquad (2.44)$$

Translational invariance, respectively the periodicity condition

$$a(n_1, \ldots, n_N) = a(n_2, \ldots, n_N, n_1 + L) \qquad (2.45)$$

gives rise to

$$\sum_{j=1}^{N} k_{p(j)} n_j + \frac{1}{2} \sum_{i<j} \theta_{p(i)p(j)} = \frac{1}{2} \sum_{i<j} \theta_{p'(i)p'(j)} - 2\pi\tilde{\lambda}_{p'(N)} + \sum_{j=2}^{N} k_{p'(j-1)} n_j + k_{p'(N)}(n_1 + L),$$

$$(2.46)$$

where $p'(i-1) = p(i)$, $i = 1, 2, \ldots, N$ and $p'(N) = p(1)$. All terms not involving $p'(N) = p(1)$ cancel, we are therefore left with N relations

$$L k_i = 2\pi\tilde{\lambda}_i + \sum_{j \neq i} \theta_{ij}, \qquad (2.47)$$

with $i = 1, \ldots, N$ and $\tilde{\lambda}_i \in \{0, 1, \ldots, L - 1\}$. We need to again find sets of Bethe quantum numbers $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N)$ which lead to solutions of the Bethe equations (2.43), (2.47). Each
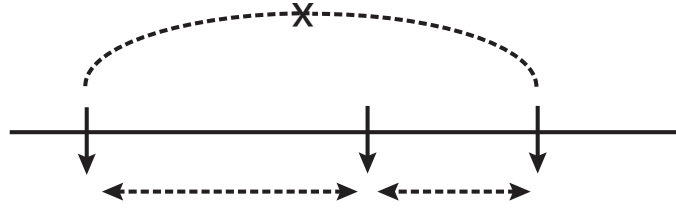
Figure 2.1: Two-body reducible chain

solution represents an eigenvector of the form Eq. (2.38) with energy (2.41) and wave number

$$k = \frac{2\pi}{L} \sum_{i=1}^{L} \tilde{\lambda}_i. \tag{2.48}$$

Similarly to the two magnon case, bound state solutions appear, this time also with three or more magnons.

In order to find a clear interpretation of the Bethe ansatz, let us rewrite the N–particle ansatz Eq. (2.38) as follows:

$$
\begin{aligned}
a(n_1, \ldots, n_N) &= \sum_{\mathcal{P} \in S_N} \exp\left(\frac{i}{2} \sum_{i<j} \theta_{p(i)p(j)}\right) \exp\left(i \sum_{j=1}^{N} k_{p(j)} n_j\right) \\
&= \sum_{\mathcal{P} \in S_N} A(k_{p(1)}, \ldots, k_{p(N)}) \exp\left(i \sum_{j=1}^{N} k_{p(j)} n_j\right).
\end{aligned}
\tag{2.49}
$$

The coefficient $A(k_{p(1)}, \ldots, k_{p(N)})$ factorizes into pair interactions:

$$A(k_{p(1)}, \ldots, k_{p(N)}) = \prod_{1 \leq i < j \leq N} e^{\frac{i}{2} \theta_{ij}}. \tag{2.50}$$

We have seen that the two-body interactions are not free, they have a non-trivial scattering matrix. The many-body collisions factorize, which means they happen as a sequence of two magnon collisions. All Bethe-solvable systems are thus *two-body reducible*. This property has to do with the fact that a spin chain is one-dimensional, so only neighboring down spins can interact directly (see Fig. 2.1). There are in fact very few two-dimensional systems that are exactly solvable.

To conclude this part, we will re-write the Bethe equations to give them a form which is more commonly used in the literature and which we will need in the last part of this lecture series. First, we introduce new variables, the so-called *rapidities* λ_i:

$$e^{ik_j} = \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}. \tag{2.51}$$

Plugging them into the periodicity condition, we get

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = \prod_{j \neq i}^{N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N. \tag{2.52}$$

This Bethe equation encodes the periodic boundary condition. In can be generalized to boundary conditions with a twist ϑ,

$$\vec{S}_{L+1} = e^{\frac{i}{2} \vartheta \sigma_z} \vec{S}_1 e^{-\frac{i}{2} \vartheta \sigma_z}: \tag{2.53}$$

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = e^{i\vartheta} \prod_{j \neq i}^N \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N. \tag{2.54}$$

The Bethe ansatz as it was presented in this lecture follows Bethe's original treatment and is referred to as the *coordinate Bethe ansatz*. It has the advantage that its physics is intuitively very clear. It can be generalized to the xxz spin chain, but not much beyond that. The so-called *algebraic* Bethe ansatz is mathematically more elegant and much more powerful. It uses concepts such as the Yang–Baxter equations, the Lax operator and the R–matrix and relies heavily on the machinery of Lie algebras and beyond. We will tackle it in the next chapter.

As a last remark, we make the non-trivial observation that the Bethe equations (2.54) describe the critical points of a potential, the so-called *Yang–Yang counting function Y*. We can rewrite the Bethe equations as

$$e^{2\pi i \varpi(\lambda)} = 1. \tag{2.55}$$

The one-form $\varpi = \sum_{j=1}^N \varpi_j(\lambda)\mathrm{d}\lambda_j$ is closed and $\varpi = \mathrm{d}Y$, with

$$Y(\lambda) = \frac{L}{2\pi} \sum_{i=1}^N \hat{x}(2\lambda_i) - \frac{1}{2\pi} \sum_{i,j=1}^N \hat{x}(\lambda_i - \lambda_j) + \sum_{j=1}^N \lambda_j \left(n_j - \frac{\vartheta}{2\pi}\right), \tag{2.56}$$

$$\hat{x}(\lambda) = \lambda \frac{i}{2} \left(\log(1 - \frac{i}{\lambda}) - \log(1 + \frac{i}{\lambda})\right) + \frac{1}{2} \log(1 + \lambda^2), \tag{2.57}$$

where the n_i are integers. The Bethe equations thus ultimately take the form

$$e^{2\pi \mathrm{d}Y(\lambda)} = 1. \tag{2.58}$$

The xxz chain and $SU_q(2)$.   Remember the Hamiltonian of the anisotropic xxz spin chain,

$$H_\Delta = -J \sum_{n=1}^L S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta(S_n^z S_{n+1}^z - \tfrac{1}{4}). \tag{2.59}$$

The anisotropy is captured by the parameter

$$\Delta = \frac{q + 1/q}{2}. \tag{2.60}$$

While $\Delta = 1$ is the isotropic case we have treated so far, the limit $\Delta = \infty$ corresponds to the one-dimensional Ising model. The xxz chain described by the Hamiltonian (2.59) can be mapped to a two-dimensional combinatorial model, the *six-vertex* or *ice-type* model.

It was first realized by Pasquier and Saleur that the xxz spin chain admits the group $SU_q(2)$ (alternative notations are $U_q[su(2)]$ and $U_q(sl_2)$) as symmetry group. This is an example of a *quantum group*, in the sense that it is a deformation by a parameter q. Quantum groups have a Hopf–algebra structure, which is also the case here.

$SU_q(2)$ is generated by S^+, S^- and $q^{\pm S^z}$ under the relations

$$q^{S^z} S^\pm q^{\pm 1} S^\pm, \qquad [S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - 1/q}. \tag{2.61}$$

These relations reduce to the ones of $SU(2)$ for $q \to 1$. For the case of spin 1/2, we find the following representations for the operators:

$$q^{S^Z} = \underbrace{q^{\sigma^3/2} \otimes \cdots \otimes q^{\sigma^3/2}}_{L}, \tag{2.62}$$

$$S^{\pm} = \sum_{i=1}^{L} S_i^{\pm} = \sum_{i=1}^{L} \underbrace{q^{\sigma^3/2} \otimes \cdots \otimes q^{\sigma^3/2}}_{i-1} \otimes \sigma_i^{\pm}/2 \otimes q^{-\sigma^3/2} \otimes \cdots \otimes q^{-\sigma^3/2}, \tag{2.63}$$

where we have the Pauli matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.64}$$

and

$$q^{\sigma^3} = \begin{pmatrix} q & 0 \\ 0 & 1/q \end{pmatrix}. \tag{2.65}$$

Since H_Δ also commutes both with S^Z and the translational operator, the Bethe ansatz works the same as for the xxx chain.

Exercise: Rederive the coordinate Bethe ansatz for H_Δ.

Literature.   This chapter follows largely [1], which itself follows Bethe's original work [2]. The first chapter of [3] is also very useful as an introduction. The supergroup structure of the xxz chain is discussed in [4].

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Chapter 3

Introduction to the Algebraic Bethe Ansatz

"Working with integrable models is a delightful pastime."

L.D. Faddeev

The algebraic Bethe ansatz (ABA) is an algebraic way of deriving the Bethe ansatz equations. It is also called the *quantum inverse scattering method* and was developed in the late seventies and early eighties by the so-called Leningrad School (proponents of which include Faddeev, Izergin, Korepin, Kulish, Reshetikhin, Sklyanin and Takhtajan).

We are about to uncover the mathematical reason for the integrability of the $\text{XXZ}_{1/2}$ spin chain, the underlying reason why Bethe's original ansatz works. As we will see, the $su(2)$–symmetry and translation invariance of the $\text{XXZ}_{1/2}$ spin chain, which have already proved instrumental in Bethe's original solution, are merely the tip of the iceberg. We are about to find an infinite dimensional symmetry (or rather L–dimensional for a finite chain of length L, which remains valid for $L \to \infty$).

For the time being, we will remain with our (by now) good old friend, the $\text{XXZ}_{1/2}$ spin chain. Recall that we have a closed chain with L sites. Its Hilbert space has the form

$$\mathcal{H} = \bigotimes_{n=1}^{L} \mathcal{H}_n, \quad \mathcal{H}_n = \mathbb{C}^2. \tag{3.1}$$

For now discounting the prefactor of $-J$ and introducing the constant $-1/4$ which sets the vacuum energy to zero, we write the Hamiltonian as

$$H = \sum_{\alpha=1}^{3} \sum_{n=1}^{L} (S_n^\alpha S_{n+1}^\alpha - \tfrac{1}{4}), \tag{3.2}$$

where the spin operators are as before $S^\alpha = \frac{1}{2}\sigma^\alpha$ and H fulfills $[H, S^\alpha] = 0$ and $S_{n+L}^\alpha = S_n^\alpha$. The fundamental object of the ABA is the *Lax operator*, a generating object that takes its name from the Lax operator used in the solution of the KdV equation. In order to define it, we must introduce the auxiliary space V which for the time being is also $\mathbb{C}^2$, and a continuous complex parameter λ, the *spectral parameter*. This parameter will allow us later on to recover the integrals of motion as coefficients of a series expansion in λ. The auxiliary space V on the other hand is needed to show that the integrals of motion commute. The Lax operator acts on $\mathcal{H}_n \otimes V$:

$$L_{n,a}(\lambda) = \lambda \mathbb{1}_n \otimes \mathbb{1}_a + i \sum_{\alpha=1}^{3} S_n^\alpha \otimes \sigma^\alpha, \tag{3.3}$$

where the labels n refer to $\mathcal{H}_n$ while the labels a refer to V. Alternatively, we can express the V–dependence of L explicitly by writing L as a 2×2 matrix:

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + i\,S_n^3 & i\,S_n^- \\ i\,S_n^+ & \lambda - i\,S_n^3 \end{pmatrix}. \tag{3.4}$$

We can write $L_{n,a}(\lambda)$ in yet another form using the fact that the operator

$$\Pi = \tfrac{1}{2}(\mathbb{1} + \sum_{\alpha=1}^{2} \sigma^\alpha \otimes \sigma^\alpha) \tag{3.5}$$

is the permutation operator on $\mathbb{C}^2 \times \mathbb{C}^2$. Since $\mathcal{H}_n$ and V are the same,

$$L_{n,a}(\lambda) = (\lambda - \tfrac{i}{2})\mathbb{1}_{n,a} + i\,\Pi_{n,a}. \tag{3.6}$$

Let us now establish the main property of the Lax operator, namely the commutation relations for its entries. In order to take the commutator, we consider two Lax operators $L_{n,a_1}(\lambda)$, $L_{n,a_2}(\lambda)$ acting on the same Hilbert space but different auxiliary spaces V_1, V_2. The product $L_{n,a_1}(\lambda)L_{n,a_2}(\lambda)$ acts on the space $\mathcal{H}_n \otimes V_1 \otimes V_2$. We now make the claim that there exists an operator $R_{a_1,a_2}(\lambda - \mu)$ in $\otimes V_1 \otimes V_2$ such that the following relation is true:

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \tag{3.7}$$

This is the fundamental commutation relation (FCR) for L, it has the form of a *Yang–Baxter equation* (YBE). The operator R, called the *R–matrix* acts as an intertwiner. The explicit expression for R is given by

$$R_{a_1,a_2}(\lambda) = \lambda \mathbb{1}_{n,a} + i\,\Pi_{n,a}. \tag{3.8}$$

We see that L and R have the same form, in fact .

**Exercise:** Verify Eq. (3.7) using the explicit expressions for L and R and the relation

$$\Pi_{n,a_1}\Pi_{n,a_2} = \Pi_{a_1,a_2}\Pi_{n,a_1} = \Pi_{n,a_2}\Pi_{2_2,a_1}. \tag{3.9}$$

We can also write the Yang–Baxter equation only in terms of R, which gives it a form more familiar from other contexts:
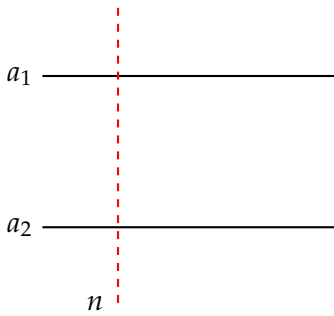
$$R_{a_1,a_2}(\tilde{\lambda} - \tilde{\mu})R_{n,a_1}(\tilde{\lambda})R_{n,a_2}(\tilde{\mu}) = R_{n,a_2}(\tilde{\mu})R_{n,a_1}(\tilde{\lambda})R_{a_1,a_2}(\tilde{\lambda} - \tilde{\mu}), \tag{3.10}$$

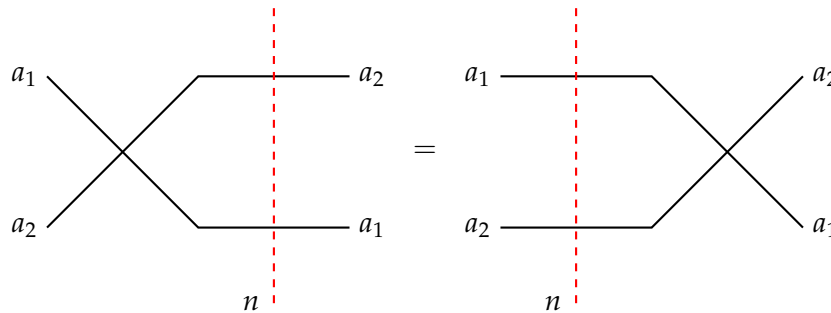where $\tilde{\lambda} = \lambda + i/2$, $\tilde{\mu} = \mu + i/2$.

We can represent the YBE diagrammatically. $L_{n,a}$ acts on two different types of space, so we depict it as the crossing of two lines of different color. R_{a_1,a_2} on the other hand acts on the same type of space and is depicted as the crossing of two a-lines.
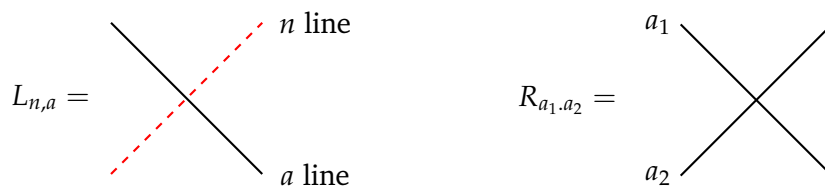
The product $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$ acts on two a-lines and one n-lines:

$$L_{n,a_1}(\lambda)L_{n,a_2}(\mu) =$$

Below, the RHS and LHS of the YBE are depicted:

The YBE also shows up in the context of knot theory and the braid group, where it states that two ways of switching the strands are equivalent. More importantly for us, the YBE has a physical interpretation in terms of scattering processes. For a scattering matrix of two on-shell particles which allows only the direct (*i.e.* the identity) and the reflection scattering processes, the multiple particle scattering factorizes into pairwise scattering processes. If you replace R by the scattering matrix S in Eq. (3.10), it encoded the fact that in the three-particle scattering, the order of the two-particle interactions does not matter. This is also easily read off from the diagrammatic representation. The statements "system X is two-particle reducible" and "system X fulfills the YBE" are therefore equivalent. We have seen in the last section that two-particle reducibility is a key property of all Bethe solvable systems, a fact that is also encoded in the fundamental commutation relation of the Lax operator.

The Lax operator also has a natural geometric interpretation as a connection along the spin chain. Consider the vector

$$\psi_n = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}, \quad \psi_n^{1,2} \in \mathcal{H}_n \text{ fermion operators} \tag{3.11}$$

in $\mathcal{H}_n \otimes V$. The Lax equation defines the parallel transport between the sites n and $n+1$:

$$\psi_{n+1} = L_n\psi_n. \tag{3.12}$$

The transport from n_1 to $n_2 + 1$ is given by the ordered product of Lax operators over all sites in between:

$$T_{n_1,a}^{n_2}(\lambda) = L_{n_2,a}(\lambda)L_{n_2-1,a}(\lambda)\ldots L_{n_1,a}(\lambda). \tag{3.13}$$

The full product over the entire chain is the monodromy around the circle:

$$T_{L,a}(\lambda) = L_{L,a}(\lambda)L_{L-1,a}(\lambda)\ldots L_{1,a}(\lambda) \in \text{End}(\mathcal{H} \otimes V). \tag{3.14}$$

$T_{L,a}(\lambda)$ describes the transport of spin around the circular chain. In the following, we will drop the label L and write T_a when referring to the monodromy matrix.

T_a acts on the Hilbert space of the full chain, just as the integrals of motion. We will see in the following that it is the generating object for spin, the Hamiltonian among others. In the following it will be convenient to express T_a as a (2×2) matrix in V:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad A, B, C, D \in \text{End}(\mathcal{H}). \tag{3.15}$$

The FCR for $T_a(\lambda)$ is given by

$$R_{a_1,a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu), \tag{3.16}$$

which has again the form of the YBE.

Exercise: Verify the FCR for $T_a(\lambda)$.

Let us now begin to extract the observables of the XXX$_{1/2}$ chain from $T_a(\lambda)$. $T_a(\lambda)$ is a polynomial in λ of order L:

$$T_a(\lambda) = \lambda^L + i\lambda^{L-1}\sum_\alpha (S^\alpha \otimes \sigma^\alpha) + \dots. \tag{3.17}$$

We see that the total spin appears in the coefficient of the second highest degree. Next, we will look for the Hamiltonian. In order to do so, we take the trace of $T_a(\lambda)$ on the auxiliary space V:

$$t(\lambda) = \text{Tr}_a[T_a(\lambda)] = A(\lambda) + D(\lambda). \tag{3.18}$$

This is the so-called *transfer matrix*. Via the FCR Eq. (3.16), we find that then traces commute for different values of the spectral parameter:

$$[t(\lambda), t(\mu)] = 0. \tag{3.19}$$

Also $t(\lambda)$ is a polynomial in λ of order L:

$$t(\lambda) = 2\lambda^L \mathbb{1} + \sum_{l=0}^{L-2} Q_l \lambda^l. \tag{3.20}$$

It produces $L-1$ operators Q_l on $\mathcal{H}$, which via the commutator of $t(\lambda)$ Eq (3.19) we find to be commuting,

$$[Q_l, Q_m] = 0. \tag{3.21}$$

In the following, we will make use of the fact that the point $\lambda = 1/2$ is special:

$$L_{n,a}(i/2) = i\,\Pi_{n,a}. \tag{3.22}$$

The relation

$$\frac{d}{d\lambda}L_{n,a}(\lambda) = \mathbb{1}_{n,a} \tag{3.23}$$

holds for any λ. Therefore

$$T_a(i/2) = i^L \Pi_{L,a}\Pi_{L-1,a}\dots\Pi_{1,a} \tag{3.24}$$

$$= i^L \Pi_{1,2}\Pi_{2,3}\dots\Pi_{L-1,L}\Pi_{L,a}. \tag{3.25}$$

Taking the trace over the auxiliary space we find

$$t(i/2) = i^L U, \tag{3.26}$$

where we defined the shift operator

$$U = \Pi_{1,2}\Pi_{2,3}\ldots\Pi_{L-1,L} \tag{3.27}$$

in $\mathcal{H}$ which simultaneously shifts all spins by one site and thus corresponds to the rotation of the chain by one site. Take the position variable X_n on site n:

$$X_n U = U X_{n-1}. \tag{3.28}$$

Since the permutation operator fulfills $\Pi^* = \Pi$ and $\Pi^2 = \mathbb{1}$, we have $U^*U = UU^* = \mathbb{1}$, *i.e.* U is unitary. Therefore

$$U^{-1}X_n U = X_{n-1}. \tag{3.29}$$

We can use U to introduce a new observable, namely the momentum P. By definition, P produces an infinitesimal shift. On the lattice, this translates to a shift by one lattice site:

$$e^{iP} = U. \tag{3.30}$$

We now proceed to expand $t(\lambda)$ around $\lambda = i/2$.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}T_a(\lambda)\big|_{\lambda=i/2} = i^{L-1}\sum_n \Pi_{L,a}\ldots\widehat{\Pi}_{n,a}\ldots\Pi_{1,a}, \tag{3.31}$$

where the hat indicates that the factor is absent. Taking the trace over V,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\big|_{\lambda=i/2} = i^{L-1}\sum_n \Pi_{1,2}\ldots\Pi_{n-1,n+1}\ldots\Pi_{L-1,L}. \tag{3.32}$$

Now we multiply by $t(\lambda)^{-1}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)t(\lambda)^{-1}\big|_{\lambda=i/2} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln t(\lambda)\big|_{\lambda=i/2} = -i\sum_n \Pi_{n,n+1}. \tag{3.33}$$

As we have seen back in Eq. (2.1), we can express the Hamiltonian in terms of the permutation operator, so we find that

$$H = \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln t(\lambda)\big|_{\lambda=i/2} - \frac{L}{2}. \tag{3.34}$$

We have thus seen that H is indeed part of the family of $L-1$ commuting operators generated by the transfer matrix. One component of the total spin, *e.g.* S^3, completes this family to a family of L commuting operators. Since the underlying classical model has L degrees of freedom, this means that it is integrable.

Bethe ansatz equations for the $\text{XXX}_{1/2}$ **spin chain.** We are now ready to connect to the main result of the last section, namely the Bethe ansatz equations. As the Hamiltonian appears as a coefficient in the expansion of the transfer matrix at $\lambda = i/2$,

$$t(\lambda) = i^L U = i^{L-1}(\lambda - i/2)U(2H + L) + \mathcal{O}((\lambda - i/2))^2, \tag{3.35}$$

the eigenvalue problem for H is solved by diagonalizing $t(\lambda) = A(\lambda) + D(\lambda)$. We will need the following relations between the quantities A, B, C, D which form the matrix representation of the monodromy matrix, see Eq. (3.15), which are derived from the FCR using explicit matrix representations:

$$[B(\lambda), B(\mu)] = 0, \tag{3.36}$$

$$A(\lambda)B(\mu) = f(\lambda - \mu)B(\mu)A(\lambda) + g(\lambda - \mu)B(\lambda)A(\mu), \tag{3.37}$$

$$D\lambda)B(\mu) = h(\lambda - \mu)B(\mu)D(\lambda) + k(\lambda - \mu)B(\lambda)D(\mu), \tag{3.38}$$

where

$$f(\lambda - \mu) = \frac{\lambda - i}{\lambda}, \qquad\qquad g(\lambda - \mu) = \frac{i}{\lambda}, \qquad\qquad (3.39)$$

$$h(\lambda - \mu) = \frac{\lambda + i}{\lambda}, \qquad\qquad k(\lambda - \mu) = -\frac{i}{\lambda}. \qquad\qquad (3.40)$$

The full set of commutation relations is symmetric under the exchange $A \leftrightarrow D$, $B \leftrightarrow C$, which corresponds to switching all the up and down spins on the chain.

$A(\lambda) + D(\lambda)$ has to be diagonal on the eigenstates, while $C(\lambda)$ and $B(\lambda)$ act as raising and lowering operators. A crucial step is to identify a highest weight state. Such a state is the reference state Ω with

$$C(\lambda)\Omega = 0. \qquad\qquad (3.41)$$

$T_a(\lambda)$ acting on Ω is thus upper triangular. This is true when $L_{n,a}$ as given in Eq. (3.4) becomes upper triangular in V_a when acting on a local state $\omega_n \in \mathcal{H}_n$:

$$L_n(\lambda)\omega_n = \begin{pmatrix} \lambda + i/2 & * \\ 0 & \lambda - i/2 \end{pmatrix} \omega_n. \qquad\qquad (3.42)$$

This is the case when $\sigma_n^+ \omega_n = 0$, so we can identify $\omega_n = |\uparrow\rangle$, and

$$\Omega = \bigotimes_{n=1}^{L} \omega_n = |\uparrow \ldots \uparrow\rangle, \qquad\qquad (3.43)$$

just as we have seen already earlier on when studying the coordinate Bethe ansatz. Acting with the monodromy matrix on Ω, we find

$$T_a(\lambda)\,\Omega = \begin{pmatrix} \alpha^L(\lambda) & * \\ 0 & \delta^L(\lambda) \end{pmatrix} \Omega, \qquad\qquad (3.44)$$

with

$$\alpha(\lambda) = \lambda + i/2, \qquad\qquad \delta(\lambda) = \lambda - i/2. \qquad\qquad (3.45)$$

We see thus that Ω is an eigenstate of $A(\lambda)$ and $D(\lambda)$ and so also of $t(\lambda)$. All other eigenvectors can be obtained from Ω by acting with the lowering operator $B(\lambda)$ on it. We will therefore be looking for eigenvectors of level N of $t(\lambda)$ of the form

$$\Phi(\{\lambda_1, \ldots, \lambda_N\}) = B(\lambda_1) \ldots B(\lambda_N)\Omega. \qquad\qquad (3.46)$$

Different orderings of the $B(\lambda_i)$ lead to the same eigenstate due to Eq. (3.36). $\Phi(\{\lambda_1, \ldots, \lambda_N\})$ is only for certain values of the $\lambda_1, \ldots, \lambda_N$ an eigenvector of $t(\lambda)$. Requiring it to be an eigenvector leads to a set of algebraic conditions on the parameters $\lambda_1, \ldots, \lambda_N$. We will now act with $A(\lambda)$ and $D(\lambda)$ on (3.46) and use the relations (3.37) and (3.38) to commute them through the $B(\lambda_i)$. The result has the form

$$\begin{pmatrix} A(\lambda) \\ D(\lambda) \end{pmatrix} B(\lambda_1) \ldots B(\lambda_N)\,\Omega = \begin{pmatrix} \prod_{k=1}^{N} f(\lambda - \lambda_k)\alpha^L(\lambda) \\ \prod_{k=1}^{N} h(\lambda - \lambda_k)\delta^L(\lambda) \end{pmatrix} B(\lambda_1) \ldots B(\lambda_N)\,\Omega$$

$$+ \sum_{k=1}^{N} \begin{pmatrix} M_k(\lambda, \{\lambda\}) \\ N_k(\lambda, \{\lambda\}) \end{pmatrix} B(\lambda_1) \ldots \widehat{B}(\lambda_k) \ldots B(\lambda_N)B(\lambda)\,\Omega, \quad (3.47)$$

where the first term already has the right form of an eigenvector of $A(\lambda)$ and $D(\lambda)$, while we lumped all the rest into the expressions M_k, N_k. Our goal is to find conditions on the $\{\lambda\}$

that will make terms of the form $\sum_{k=1}^N (M_k(\lambda, \{\lambda\}) + N_k(\lambda, \{\lambda\})) B(\lambda_1) \ldots \widehat{B}(\lambda_k) \ldots B(\lambda_N) B(\lambda) \Omega$ disappear. It is easy to determine M_1, N_1 from the relations (3.37) and (3.38), *e.g.*

$$M_1(\lambda_1, \{\lambda\}) = g(\lambda - \lambda_1) \prod_{k=2}^N f(\lambda - \lambda_1) \alpha^L(\lambda_1). \tag{3.48}$$

From the commutation relation (3.38), we learn that we can simply substitute λ_1 with any λ_k, so

$$M_j(\lambda_1, \{\lambda\}) = g(\lambda - \lambda_j) \prod_{k \neq j}^N f(\lambda_j - \lambda_k) \alpha^L(\lambda_j). \tag{3.49}$$

Similarly,

$$N_j(\lambda_1, \{\lambda\}) = k(\lambda - \lambda_j) \prod_{k \neq j}^N h(\lambda_j - \lambda_k) \delta^L(\lambda_j). \tag{3.50}$$

Note that $g(\lambda - \lambda_j) = -k(\lambda - \lambda_j)$. We find that

$$(A(\lambda) + D(\lambda)) \Phi(\{\lambda\}) = \Lambda(\lambda, \{\lambda\}) \Phi(\{\lambda\}) \tag{3.51}$$

for

$$\Lambda(\lambda, \{\lambda\}) = \alpha^L(\lambda) \prod_{j=1}^N f(\lambda - \lambda_j) + \delta^L(\lambda) \prod_{j=1}^N h(\lambda - \lambda_j) \tag{3.52}$$

if the set $\{\lambda\}$ satisfies the equation

$$\prod_{k \neq j}^N f(\lambda_j - \lambda_k) \alpha^L(\lambda_j) = \prod_{k \neq j}^N h(\lambda_j - \lambda_k) \delta^L(\lambda_j). \tag{3.53}$$

Using the explicit expressions (3.39) and (3.45), this is the Bethe equation

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^L = \prod_{j \neq k}^N \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}. \tag{3.54}$$

In other words, we have recovered by algebraic means the exact expression of Eq. (2.52), which is naturally the way it should be, given that we have studied the same system. The set $\{\lambda\}$ is called the *Bethe roots* and the expression (3.46) the Bethe vector.

Let us now study the properties of the eigenvectors (3.46), such as their spin, momentum and energy.

Spin. Taking $\mu \to \infty$ in the YBE (3.7) and representing the monodromy in terms of spin operators using Eq. (3.3), we find

$$[T_a(\lambda), \tfrac{1}{2}\sigma^\alpha + S^\alpha] = 0, \tag{3.55}$$

which expresses the $su(2)$ invariance of $T_a(\lambda)$ in $\mathcal{H} \otimes V$. In particular, we have

$$[S^3, B] = -B, \qquad\qquad [S^+, B] = A - D. \tag{3.56}$$

We know that for the reference state Ω,

$$S^+ \Omega = -0, \qquad\qquad S^3 \Omega = \tfrac{L}{2} \Omega. \tag{3.57}$$

Ω is thus a highest weight state for the spin operator S^α. Applying the relations Eq. (3.56) to the Bethe vectors, we find

$$S^3 \Phi(\{\lambda\}) = (\tfrac{L}{2} - N)\Phi(\{\lambda\}), \tag{3.58}$$

$$S^+ \Phi(\{\lambda\}) = \sum_j B(\lambda_1)\dots B(\lambda_{j-1})(A(\lambda_j) - D(\lambda_j))B(\lambda_{j+1})\dots B(\lambda_N)\Omega \tag{3.59}$$

$$= \sum_k O_k(\{\lambda\}) B(\lambda_1)\dots \widehat{B}(\lambda_k) \dots B(\lambda_{j+1})\dots B(\lambda_N)\Omega. \tag{3.60}$$

Along the same lines as we have derived the Bethe equations, one can show that the coefficients O_k all vanish if the $\{\lambda\}$ fulfill the Bethe equations (3.54). The Bethe vectors are thus all highest weight states of the spin operator.

Since the S^3 eigenvalue of the highest weight state must be non-negative, we must have $N \leq L/2$. The spectrum of H is degenerate under the exchange of all spin up and spin down states, so effectively the whole range is covered.

Momentum. Recall that $t(i/2) = i^L U$. Using Eq. (3.52), we find

$$\Lambda(i/2, \{\lambda\}) = i^L \prod_{j=1}^N \frac{\lambda_j + i/2}{\lambda_j - i/2}. \tag{3.61}$$

Taking the logarithm for the momentum,

$$P \Phi(\{\lambda\}) = -i \sum_{j=1}^N \ln \frac{\lambda_j + i/2}{\lambda_j - i/2} \Phi(\{\lambda\}), \tag{3.62}$$

where we can define

$$p(\lambda_j) = -i \ln \frac{\lambda_j + i/2}{\lambda_j - i/2}. \tag{3.63}$$

We see that the momentum is additive and that each λ_j has momentum $p(\lambda_j)$.

Energy. The same is true for the energy. Using the expression for H we found in Eq. (3.34), we find

$$H \Phi(\{\lambda\}) = \sum_{j=1}^N \epsilon(\lambda_j)\Phi(\{\lambda\}), \tag{3.64}$$

with

$$\epsilon(\lambda_j) = -\frac{1}{2}\frac{1}{\lambda^2 + 1/4}. \tag{3.65}$$

We see thus that it makes sense to use a *quasiparticle interpretation* for the spectrum of observables on the Bethe vectors. Each quasiparticle is created by $B(\lambda)$, diminishes the spin S^3 by one, has momentum $p(\lambda)$ and energy $\epsilon(\lambda)$. Of course these are again the magnons we have encountered in the discussion of Bethe's original results. As we have the relation

$$\epsilon(\lambda) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} p(\lambda), \tag{3.66}$$

the λ can be interpreted as the rapidity of the quasiparticle.

The Hamiltonian in Eq. (3.2) corresponds, unlike the case we studied in the last section, to the anti-ferromagnetic case, where Ω is not the ground state. If we take instead $-H$, we are back in the ferromagnetic case, as before.

We have thus come to a full circle, having recovered by algebraic means all the physical properties of the $\text{XXX}_{1/2}$ chain we had studied via the more physically transparent coordinate Bethe ansatz.

Literature. This chapter largely follows lecture notes of Faddeev [1], with some extra padding here and there.

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Chapter 4

General Spin Chains via the Algebraic Bethe Ansatz

One of the main arguments for preferring the algebraic Bethe ansatz over the original one is that it is applicable to a range of more general spin chains. We will study in the following two generalizations.

The xxx$_s$ Chain. One obvious generalization is to study the xxx model, but for general spin s. Here, the Hilbert space is $\mathcal{H} = \mathbb{C}^{2s+1}$. Naively, one is tempted to just rewrite the Hamiltonian Eq. (3.2),

$$H = \sum_{\alpha=1}^{3} \sum_{n=1}^{L} (S_n^\alpha S_{n+1}^\alpha - \tfrac{1}{4}), \tag{4.1}$$

where we take S_n^α to be in the representation of spin s. The problem with this naive approach is, that the Hamiltonian (4.1) is not integrable!

Armed with the knowledge of the last section, we try instead via the ABA. The first naive attempt involves constructing a Lax operator on $\mathcal{H}_n \otimes V = \mathbb{C}^{2s+1} \otimes \mathbb{C}^2$. To a certain degree, this works, as the derivation of the BAE works along the same lines as before: the Lax operator

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + i\,S_n^3 & i\,S_n^- \\ i\,S_n^+ & \lambda - i\,S_n^3 \end{pmatrix} \tag{4.2}$$

satisfies the YBE (3.7) with the same R–matrix as before, $R_{a_1,a_2}(\lambda) = L_{a_1,a_2}(\lambda + i/2)$. We can define the monodromy and the reference vector $\Omega = \otimes\,\omega_n$, where ω_n is now the highest weight state in $\mathbb{C}^{2s+1}$. Also the Bethe vectors are formally the same as before, and we find the BAE for the roots $\{\lambda\}$,

$$\left(\frac{\lambda_j + is}{\lambda_j - is} \right)^L = \prod_{j \neq k}^{N} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}. \tag{4.3}$$

This is a natural and obvious generalization of Eq. (3.54). The generalization of the argument to extract H from t however does not work, as we cannot express the Lax operator via a permutation operator as in Eq. (3.6), since now $\mathcal{H}_n$ and V are not the same. We must therefore adopt a more general approach and construct a Lax operator for $\mathcal{H}_n$ and V isomorphic.

Take an abstract algebra $\mathcal{A}$, which is defined via the ternary relation on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{4.4}$$

where $\mathcal{R}$ is a universal R–matrix acting on $\mathcal{A} \otimes \mathcal{A}$ and the labels indicate on which of the three factors of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ $\mathcal{R}$ acts:

$$\mathcal{R}_{12} = \mathcal{R} \otimes \mathbb{1}, \qquad\qquad \mathcal{R}_{23} = \mathbb{1} \otimes \mathcal{R}, \qquad\qquad (4.5)$$

etc. $\mathcal{A}$ has a family of representations $\rho(a, \lambda)$ parametrized by a discrete label a and a continuous label λ. For the XXX model, $\mathcal{A}$ is the so-called *Yangian*, named by Drinfeld after C.N. Yang. The Yangian is another example of a quantum group, it has the structure of an infinite-dimensional Hopf algebra. Most generally speaking, it is a deformation of the universal enveloping algebra $U(a[z])$ of the semi-simple Lie algebra of polynomial loops of the semi-simple Lie algebra a. We will however not work at this level of generality. The definition via the YBE (4.4) works for the Lie algebra $a = gl(n)$ and is thus sufficient for our needs.

Concretely, we can obtain the Lax operators via the evaluation representation of the universal R-matrix:

$$L_{n,a}(\lambda, \mu) = (\rho(a, \lambda) \otimes \rho(n, \mu))\mathcal{R} = R^{a,n}(\lambda - \mu), \qquad\qquad (4.6)$$

where $\rho(a, \lambda)$ is a representation of the Yangian of $su(2)$ and the discrete label $a = 0, 1/2, 1, \ldots$ is the spin label. We can recover our original YBE Eq. (3.7) for spin $1/2$ from Eq. (4.4) by applying the representations $\rho(1/2, \lambda) \otimes \rho(1/2, \mu) \otimes \rho(s, \sigma)$, setting $\sigma = 0$ and making the identifications

$$R_{a_1, a_2}(\lambda) = R^{1/2, 1/2}(\lambda), \qquad\qquad L_{n, a_1}(\lambda) = R^{1/2, s}(\lambda). \qquad\qquad (4.7)$$

In order to study the general case of spin s, it is most convenient to cast the YBE Eq. (4.4) via permutations into the form

$$\mathcal{R}_{12}\mathcal{R}_{32}\mathcal{R}_{31} = \mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}_{12}. \qquad\qquad (4.8)$$

To this YBE, we apply the representation $\rho(s_1, \lambda) \otimes \rho(s_2, \mu) \otimes \rho(1/2, \sigma)$, where s_1 labels the local Hilbert space $\mathcal{H}_n$ and s_2 labels the auxiliary space V. This leads to the relation

$$R^{s_1, s_2}(\lambda - \mu)R^{1/2, s_2}(\sigma - \mu)R^{1/2, s_1}(\sigma - \lambda) = R^{1/2, s_1}(\sigma - \lambda)R^{1/2, s_2}(\sigma - \mu)R^{s_1, s_2}(\lambda - \mu).$$
$$(4.9)$$

For the case we are interested in where $\mathcal{H}_n$ and V have equal dimension, namely $s_1 = s_2$, we can construct a new fundamental Lax operator $L_{n,f}(\lambda)$ from the YBE above. To do so, let us denote the two sets of spin operators labeled by s_1 and s_2 by S^α and T^α, and their Lax operators as

$$L_S(\lambda) = \lambda + i(S, \sigma), \qquad\qquad L_T(\lambda) = \lambda + i(T, \sigma), \qquad\qquad (4.10)$$

where $(S, \sigma) = \sum_\alpha S^\alpha \sigma^\alpha$ and likewise for T, along the lines of the spin $1/2$ case. We will search for $R^{s_1, s_2}(\lambda)$ of the form

$$R^{s_1, s_2}(\lambda) = \Pi_{s_1, s_2} r((S, T), \lambda), \qquad\qquad (4.11)$$

where Π_{s_1, s_2} is the permutation operator in $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$, $\Pi(S, \sigma)\Pi = (T, \sigma)$, and r is an operator which depends on the Casimir

$$C = (S, T) = \sum_\alpha S^\alpha T^\alpha \qquad\qquad (4.12)$$

and λ. Plugging L_S, L_T and R^{s_1, s_2} into Eq. (4.9), we can rewrite it as

$$(\lambda - i(T, \sigma))(\mu - i(S, \sigma))r(\lambda - \mu) = r(\lambda - \mu)(\mu - i(T, \sigma))(\lambda - i(S, \sigma)). \qquad (4.13)$$

Using the property of the Pauli matrices

$$(T, \sigma)(S, \sigma) = (T, S) + i((S \times T), \sigma), \tag{4.14}$$

where $(S \times T)^\alpha = \epsilon_{\alpha\beta\gamma} S^\beta T^\gamma$, and the fact that the Casimir commutes with everything, we can turn Eq. (4.14) into

$$(\lambda S^\alpha + (S \times T)^\alpha) r(\lambda) = r(\lambda)(\lambda T^\alpha + (S \times T)^\alpha). \tag{4.15}$$

It is enough to consider one out of the three equations above, *e.g.* the combination

$$(\lambda S^+ + i(T^3 S^+ - S^3 T^+)) r(\lambda) = r(\lambda)(\lambda T^+ + i(T^3 S^+ - S^3 T^+)). \tag{4.16}$$

We will switch to the variable J instead of (S, T) and use the irreducibility of the representations of S and T:

$$(S + T)^2 = S^2 + T^2 + 2(S, T) = 2s(s+1) + 2(S, T) = J(J+1), \tag{4.17}$$

where J has eigenvalue j in each irreducible representation D_j in the Clebsch-Gordan decomposition

$$D_s \otimes D_s = \sum_{j=0}^{2s} D_j. \tag{4.18}$$

Remember that we are doing all of this to find an operator r in order to construct R^{s_1, s_2} in the form Eq. (4.11). To find r as a function of J, we use Eq. (4.16) in the subspace of highest weights in each D_j, *i.e*

$$T^+ + S^+ = 0. \tag{4.19}$$

We can do this because

$$[T^3 S^+ - S^3 T^+, T^+ + S^+] = 0, \tag{4.20}$$

which means that the combination $T^3 S^+ - S^3 T^+$ appearing in Eq. (4.16) does not take you out of the subspace of highest weights. In this subspace, $J = S^3 + T^3$, so Eq. (4.16) reduces to

$$(\lambda S^+ + i J S^+) r(\lambda, J) = r(\lambda, J)(-\lambda S^+ + i J S^+). \tag{4.21}$$

Using $S^+ J = (J-1)S^+$, we find

$$(\lambda + iJ) r(\lambda, J-1) = r(\lambda, J)(-\lambda + iJS), \tag{4.22}$$

which is a functional equation for $r(\lambda, J)$. It is solved by

$$r(\lambda, J) = \frac{\Gamma(J + 1 + i\lambda)}{\Gamma(J + 1 - i\lambda)}, \tag{4.23}$$

where the Γ–function appears (remember $\Gamma(n+1) = n!$). It is normalized such that $r(0, J) = \mathbb{1}$ and $r(-\lambda, J) r(\lambda, J) = \mathbb{1}$. In the case we are interested in, $J = 0, 1, \ldots, 2s$, but in principle $r(\lambda, J)$ can be used in all generality with $s \in \mathbb{C}$. After all of this, we can now plug Eq. (4.23) into Eq. (4.11) and rewrite the FCR as

$$R_{f_1, f_2}(\lambda - \mu) L_{n, f_1}(\lambda) L_{n, f_2}(\mu) = L_{n, f_2}(\mu) L_{n, f_1}(\lambda) R_{f_1, f_2}(\lambda - \mu). \tag{4.24}$$

We finally have an expression where the auxiliary spaces V_1, V_2 are isomorphic to the local Hilbert space $\mathcal{H}_n = \mathbb{C}^{2s+1}$. From here on, we can now reproduce the reasoning used for the case $s = 1/2$. We can define the momodromy

$$T_f(\lambda) = L_{L,f}(\lambda) L_{L-1,f}(\lambda) \ldots L_{1,f}(\lambda) \tag{4.25}$$

and the transfer matrix

$$t_f(\lambda) = \text{Tr}_f T_f(\lambda). \tag{4.26}$$

We find again

$$[t_f(\lambda), t_f(\mu)] = 0. \tag{4.27}$$

Using Eq. (4.4) with $s_1 = s_2 = s$, $R^{1/2,s}(\lambda - \mu)$ as R–matrix and $R^{1/2,s}(\mu)$, $R^{s,s}(\lambda)$ as Lax operators, we find commutativity also for the families $t_f(\lambda)$ and $t_a(\lambda)$:

$$[t_f(\lambda), t_a(\mu)] = 0. \tag{4.28}$$

To derive the BAE, we use $t_a(\lambda)$ (*i.e* the case $s = 1/2$), while for the observables, we use $t_f(\lambda)$:

$$t_f(0) = U = e^{iP}, \tag{4.29}$$

$$H = i\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln t_f(\lambda)|_{\lambda=0} = \sum_{n=1}^{L} H_{n,n+1}, \tag{4.30}$$

$$H_{n,n+1} = i\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln r(J,\lambda)|_{\lambda=0}, \tag{4.31}$$

where J is constructed via the local spins S_n^α, S_{n+1}^α:

$$J(J+1) = 2\sum_\alpha (S_n^\alpha S_{n+1}^\alpha) + 2s(s+1). \tag{4.32}$$

We get thus

$$H_{n,n+1} = -2\psi(J+1), \tag{4.33}$$

where ψ is the logarithmic derivative of the Γ–function. For $n \in \mathbb{N}$, $\psi(1+n) = \sum_{k=1}^{N} \frac{1}{k} - \gamma$, where γ is the Euler constant. $H_{n,n+1}$ can be expressed in terms of a polynomial in $C_{n,n+1} = \sum_\alpha S_n^\alpha S_{n+1}^\alpha$:

$$H_{n,n+1} = \sum_{j=1}^{2s} c_k (C_{n,n+1})^k = f_{2s}(C_{n,n+1}), \text{ with} \tag{4.34}$$

$$f_{2s}(x) = \sum_{j=1}^{2s} \left(\sum_{k=1}^{j} \frac{1}{k} - \gamma \right) \prod_{l=0}^{2s} \frac{x - x_l}{x_j - x_l}, \text{ where} \tag{4.35}$$

$$x_l = \tfrac{1}{2}(l(l+1) - 2s(s+1)). \tag{4.36}$$

Let us write this down explicitly for the case of $s = 1$. Here, $c_1 = -c_2$, so

$$H = \sum_{\alpha,n} (S_n^\alpha S_{n+1}^\alpha - (S_n^\alpha S_{n+1}^\alpha)^2). \tag{4.37}$$

This generalization from the spin 1/2 Hamiltonian one could not have easily guessed! Last, we give the eigenvalues of the fundamental transfer matrix $t_f(\lambda)$ on the Bethe vector $\Phi(\{\lambda\})$:

$$\Lambda_f(\lambda, \{\lambda\}) = \sum_{-s}^{s} \alpha_m(\lambda)^L \prod_{k=1}^{l} c_m(\lambda - \lambda_k). \tag{4.38}$$

To extract the energy and the momentum of the quasiparticles, it is enough to know

$$\begin{cases} \alpha_m(0) = 0, & m = -s, -s+1, \ldots, s-1, \\ \alpha_s(0) = 1, \end{cases} \tag{4.39}$$

and

$$c_s(\lambda) = \frac{\lambda - is}{\lambda + is}.$$ (4.40)

With this, we get

$$p(\lambda) = -i \ln \frac{\lambda - is}{\lambda + is}, \qquad \epsilon(\lambda) = -\frac{s}{\lambda^2 + s^2} \quad \text{and} \qquad (4.41)$$

$$\epsilon(\lambda) = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} p(\lambda).$$ (4.42)

We see that the energy and momentum of the magnon generalize in a straightforward way from the case of spin 1/2, while this is not at all the case for the Hamiltonian. The generalization of the xxx spin chain to spin s is one of the important achievements of the ABA.

Spin chains with higher rank symmetry algebra. Another obvious way in which the xxx spin chain can be generalized is by choosing a symmetry algebra with rank $r > 1$, *e.g.* $gl(n)$, where $r = n - 1$. In this case, the *nested Bethe ansatz* (Kulish, Resetikhin) must be applied. It works by reducing the rank r case to the one of rank $r - 1$, and so on, until we arrive again at rank one. Given the time constraints of this lecture series, I will merely sketch the method to give the reader a general idea of how it works.

For each of the $r - 1$ steps, a new monodromy and transfer matrix are constructed. In the first step, we decompose $T(\lambda)$ as

$$T(\lambda) = \begin{pmatrix} t_{11}(\lambda) & B^{(1)}(\lambda) \\ C^{(1)}(\lambda) & T^{(2)}(\lambda) \end{pmatrix} \in \mathrm{End}(\mathbb{C}^r),$$ (4.43)

where $B^{(1)}$ is a row vector in $\mathbb{C}^r$, $C^{(1)}$ is a column vector in $\mathbb{C}^r$, and $T^{(2)}(\lambda)$ is a matrix in $\mathrm{End}(\mathbb{C}^{r-1})$. $T^{(2)}(\lambda)$ gets decomposed in the same way in the next step. Each step gives rise to a separate Bethe vector and a set of rapidities $\lambda_j^{(a)}$ $j = 1, \dots, N_a$ $a = 1, \dots, r$. We start with an $\mathcal{A}_r$ chain with L sites, where N_1 is the number of pseudo-excitations. In the next step, we get an $\mathcal{A}_{r-1}$ with $L + N_1$ sites. The additional N_1 sites correspond to fundamental representations of $\mathcal{A}_{r-1}$. Now we need to find the eigenvectors of the $\mathcal{A}_{r-1}$ by diagonalizing the new transfer matrix. A new highest weight state needs to be identified, etc. This leads now to an $\mathcal{A}_{r-2}$ chain with $L + N_1 + N_2$ sites, and so on. The rank r chain differs from the case we have studied so far in that

- it has r different particle species, with particle numbers $N_1, \dots, N_r$.

- for each species a, there is an effective length of the chain L_a, $a = 1, \dots, r$.

- each particle species has its own set of rapidities $\lambda_j^{(a)}$

- each particle species has its own twist parameter $\vartheta^{(a)}$ for the boundary conditions.

The simple spin chain interpretation we had for the rank one case thus gets somewhat stretched by this generalization.

Let us end by giving the nested Bethe equations:

$$\prod_{n=1}^{L_a} \frac{\lambda_i + is_n^{(a)}}{\lambda_i - is_n^{(a)}} = e^{i\vartheta^{(a)}} \prod_{b=1}^{r} \prod_{\substack{j=1 \\ (i,a) \neq (j,b)}}^{N_a} \frac{\lambda_i^{(a)} - \lambda_j^{(b)} + \frac{i}{2}C^{ab}}{\lambda_i^{(a)} - \lambda_j^{(b)} - \frac{i}{2}C^{ab}},$$ (4.44)

where $i = 1, \dots, N_a$, the spin operators are realized as generators of $\mathcal{A}_r$ and C^{ab} is the Cartan matrix of $\mathcal{A}_r$.

Literature. This chapter largely follows lecture notes of Faddeev [1]. The nested Bethe ansatz was first discussed in [2].

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Chapter 5

Relations between Spin Chains and Supersymmetric Gauge Theories

Gauge theories are the foundation of our understanding of nature. Of the fundamental interactions, the electroweak force and QCD (strong interaction) are described by quantum gauge theories. Understanding gauge theories as well as possible is a top priority. Despite decades of research, there are still open problems remaining, in particular regarding their non-perturbative behavior and confinement. One way of rendering quantum gauge theories more tractable is to introduce supersymmetry. We will be using supersymmetry as a tool to gain insights into gauge theories, as a kind of *laboratory* for studying them. Supersymmetry constrains a theory and makes it well-behaved. It has a number of desirable mathematical properties, such as *e.g.* non-renormalisation theorems and protection of certain quantities from quantum corrections. The more supersymmetry a theory has, the more constrained it is, but a the same time, the less realistic it is from a phenomenological point of view (*e.g.* $\mathcal{N} = 4$ super Yang–Mills theory).

In recent years, $\mathcal{N} = 2$ gauge theories have been a focus of interest. Seiberg and Witten (1994) showed that $\mathcal{N} = 2$ SYM theory can be solved completely at the quantum level. It is possible to construct an exact low energy Lagrangian and the exact spectrum of BPS states. It displays moreover a strong/weak duality and has a rich algebraic structure surviving quantum corrections.

In the following, we will be particularly interested in *deformations* of supersymmetric gauge theories that preserve some of the supersymmetry and in particular preserve its useful properties. There will be two types of deformation of relevance:

- mass deformations (*e.g.* twisted mass deformations in 2D).

- Ω–type deformations.

In the Ω–deformation, deformation parameters ϵ_i are introduced which break Poincaré symmetry. It was introduced by Nekrasov (2004) as a calculational device for a localization calculation of the instanton sum of $\mathcal{N} = 2$ SYM. However, it is also interesting to study Ω–deformed gauge theories in their own right.

We will see in the following that these deformed gauge theories are intimately connected with integrable systems.

The relations between integrable models and supersymmetric gauge theories are a very interesting subject and active research topic. There are several examples of these connections, *e.g.*

- *2d gauge/Bethe correspondence*: $\mathcal{N} = (2,2)$ gauge theories in 2d are related to Bethe-solvable spin chains.

- *4d gauge/Bethe correspondence*: Ω–deformed $\mathcal{N} = 2$ supersymmetric gauge theories are related to quantum integrable models.

- *Alday–Gaiotto-Tachikawa (AGT) correspondence*: Ω–deformed super Yang–Mills theory in 4d is related to Liouville theory.

In these lectures, we will concentrate on the 2d gauge/Bethe correspondence (Nekrasov–Shatashvili). We will match the parameters of a spin chain to those of $\mathcal{N} = (2,2)$ supersymmetric gauge theories. It will turn out that the full (Bethe) spectrum of the spin chain corresponds one-to-one to the supersymmetric ground states of the corresponding gauge theories. We have acquired the necessary knowledge on the spin chain side of the correspondence:

Spin Chain:	**Supersymmetric gauge theories:**
• parameters of a general spin chain	• $\mathcal{N} = (2,2)$ gauge theories
• Bethe ansatz equations	• twisted mass deformation
• Yang–Yang counting function	• low energy effective action, in particular the effective twisted superpotential
	• equation for the ground states

Before we can discuss the relations between the two sides, we need to familiarize ourselves with the necessary concepts in 2d supersymmetric gauge theory. In order to do so, we follow closely sections 12.1, 12.2, 15.2 and 15.5 of the book *Mirror Symmetry* by K. Hori et al [1].[1]

Gauge/Bethe correspondence for the $\text{xxx}_{1/2}$ **spin chain.**   Recall the Bethe ansatz equation for the N–magnon sector, Eq. (2.52):

$$\left(\frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad i = 1, \ldots, N. \tag{5.1}$$

We have seen that it is expressed equivalently by

$$e^{2\pi \mathrm{d}Y(\lambda)} = 1, \tag{5.2}$$

where Y us the Yang–Yang function, given explicitly by

$$Y(\lambda) = \frac{L}{2\pi} \sum_{i=1}^{N} (\lambda_i - i/2)(\log(\lambda_i - i/2) - 1) - (\lambda_i + i/2)(\log(-\lambda_i - i/2) - 1)$$
$$- \frac{1}{2\pi} \sum_{i,j=1}^{N} (\lambda_i - \lambda_j + i)(\log(\lambda_i - \lambda_j + i) - 1) + \sum_{j=1}^{N} \lambda_j \left(n_j - \frac{\vartheta}{2\pi}\right). \tag{5.3}$$

This result we now want to compare with the $\mathcal{N} = (2,2)$ gauge theory with gauge group $U(\widetilde{N})$, one adjoint mass $\widetilde{m}^{\mathbf{adj}}$ and $\widetilde{L}$ fundamental and anti-fundamental fields Q_i, $\overline{Q}_i$ with

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[1]This actually takes a good two lectures to cover, but as I cannot really improve on the book, I see little gain in re-typing the material here.
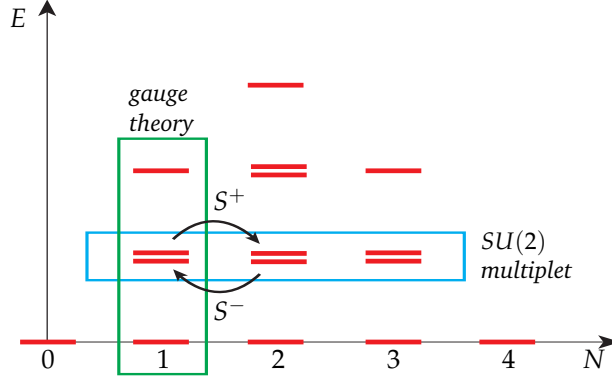
Figure 5.1: The $su(2)$ symmetry on the $\text{XXX}_{1/2}$ spin chain for $L = 4$. The horizontal arrows show the action of the $S^{\pm}$ operators, changing the magnon number N, preserving the energy. The spectrum can be organized into multiplets of $su(2)$ (horizontal box) or by magnon number (vertical box).

twisted masses $\widetilde{m}^{\mathbf{f}}$, $\widetilde{m}^{\mathbf{\bar{f}}}$. The effective twisted superpotential for this theory is

$$
\begin{aligned}
\widetilde{W}_{\mathbf{eff}} = \frac{1}{2\pi} \sum_{i=1}^{\widetilde{N}} \sum_{k=1}^{\widetilde{L}} (\sigma_i + \widetilde{m}_k^{\mathbf{f}})(\log(\sigma_i + \widetilde{m}_k^{\mathbf{f}}) - 1) - (\sigma_i + \widetilde{m}_k^{\mathbf{\bar{f}}})(\log(-\sigma_i + \widetilde{m}_k^{\mathbf{\bar{f}}}) - 1) \\
- \frac{1}{2\pi} \sum_{i,j=1}^{\widetilde{N}} (\sigma_i - \sigma_j + \widetilde{m}^{\mathbf{adj}})(\log(\sigma_i - \sigma_j + \widetilde{m}^{\mathbf{adj}}) - 1) - it \sum_{j=1}^{\widetilde{N}} \sigma_j,
\end{aligned}
\tag{5.4}
$$

where the first term comes from the fundamental fields, the second from the anti-fundamental fields, and the third from the adjoint fields. Comparing with Eq. (5.3), we find the two expressions to be the same with the following identifications:

$$
\sigma_i = \lambda_i, \qquad\qquad\qquad \widetilde{N} = N, \tag{5.5}
$$
$$
\widetilde{m}_k^{\mathbf{f}} = -i/2, \qquad\qquad\qquad \widetilde{m}_k^{\mathbf{\bar{f}}} = -i/2 \tag{5.6}
$$
$$
\widetilde{m}^{\mathbf{adj}} = i, \qquad\qquad\qquad \widetilde{L} = L, \tag{5.7}
$$
$$
t = \tfrac{1}{2\pi}\vartheta + in. \tag{5.8}
$$

The magnon number N corresponds to the number of colors of the $U(N)$ gauge theory, while the number of flavors corresponds to the length of the spin chain. Since we can identify Y and $\widetilde{W}_{\mathbf{eff}}$, the equations for the spectrum of the N–magnon sector and for the ground states of the $U(N)$ supersymmetric gauge theory are identified as well. The supersymmetric ground states and the N–particle Bethe states are in one-to-one correspondence, this is the main statement of the gauge/Bethe correspondence..

We have seen that in order to obtain the full spectrum of the spin chain, we must solve the Bethe equations for all magnon subsectors, $N = 0, 1, \ldots, L$. Taking the correspondence seriously, we should thus also consider gauge theories with different numbers of colors together, *i.e.* $U(1), \ldots, U(L)$. Let us consider the action of the symmetry group of the integrable model. For concreteness, we take an $\text{XXX}_{1/2}$ spin chain of length $L = 4$, see Figure 5.1. We see that the $S^{\pm}$ operators of $su(2)$ act horizontally between states with different N, preserving the energy. The full spectrum of the spin chain is thus organized horizontally into $su(2)$ multiplets. The gauge/Bethe correspondence, on the other hand, identifies states in an N–magnon subsector with the ground states of a gauge theory, slicing the spectrum up vertically. The action of $su(2)$ is thus a symmetry between gauge theories with different numbers of colors. This can bee seen with an obvious example: for the spin
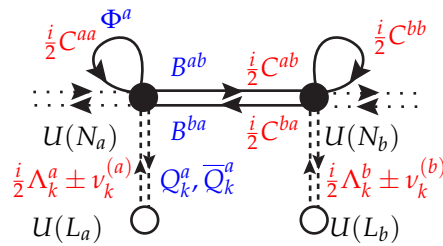
Figure 5.2: Example quiver diagram for the Gauge/Bethe correspondence. Gauge groups are labeled in black, matter fields in blue, the corresponding twisted masses in red.

chain, the physics is the same if we use the reference state $|\Omega\rangle = |\uparrow \ldots \uparrow\rangle$ or instead $|\downarrow \ldots \downarrow\rangle$. The sector with N down spins starting with $|\Omega\rangle$ and the one with $L - N$ up spins starting from the reference state $|\downarrow \ldots \downarrow\rangle$ are the same. The spectrum of the spin chain has therefore a manifest $N, L - N$ symmetry. This equivalence is reflected on the gauge theory side as the *Grassmannian duality*. The vacuum manifold of the low energy effective gauge theory corresponding to the $\text{XXX}_{1/2}$ chain is the cotangent bundle of the Grassmannian $T^*\text{Gr}(N, L)$, where the Grassmannian is the collection of all linear subspaces of dimension N of a vector space of dimension L:

$$\text{Gr}(N, L) = \{W \subset \mathbb{C}^L | \dim W = N\}, \tag{5.9}$$

$$T^*\text{Gr}(N, L) = \{(X, W), W \in \text{Gr}(N, L), X \in \text{End}(\mathbb{C}^L) | X(\mathbb{C}^L) \subset W, X(W) = 0\}. \tag{5.10}$$

The Grassmannian duality states that there is an isomorphism between $\text{Gr}(N, L)$ and $\text{Gr}(L - N, L)$, thus linking the ground states of the low energy $U(N)$ and $U(L - N)$ gauge theories.

The integrable structure of the spin chain remains hidden on the gauge theory side of the correspondence as long as the gauge theories with different numbers of colors are considered separately. A mathematical framework that unifies these gauge theories in a meaningful way is Ginzburg's *geometric representation theory*.

We have studied only the simplest example of the correspondence involving the $\text{XXX}_{1/2}$ spin chain, but the scope of the gauge/Bethe correspondence is much larger. So let us have a quick look at the general dictionary between gauge theory and spin chain parameters, see Table 5.1.

In general, we are dealing with a quiver gauge theory, which can be summarized by a graph, see Fig. 5.2. The black nodes represent gauge groups, arrows between the nodes correspond to bifundamental fields, arrows from a node to itself indicate adjoint fields, white nodes represent flavor groups and the dashed arrows between flavor and gauge nodes represent fundamental and anti-fundamental fields.

Using Table 5.1, the values of the twisted masses that we recovered from the correspondence can be traced back to the Cartan matrix of $su(2)$,

$$C^{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{5.11}$$

the fact that we had the fundamental representation at every site, so $\Lambda_k = 1$, and the absence of inhomogeneities.

gauge theory			integrable model
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a–th node	$U(N_a)$	N_a	number of particles of species a
effective twisted superpotential	$\widetilde{W}_{\mathrm{eff}}(\sigma)$	$Y(\lambda)$	Yang–Yang function
equation for the vacua	$e^{2\pi\,\mathrm{d}\,\widetilde{W}_{\mathrm{eff}}} = 1$	$e^{2\pi i\,\mathrm{d}\,Y} = 1$	Bethe ansatz equation
flavor group at node a	$U(L_a)$	L_a	effective length for the species a
lowest component of the twisted chiral superfield	$\sigma_i^{(a)}$	$\lambda_i^{(a)}$	rapidity
twisted mass of the fundamental field	$\widetilde{m}_k^{\mathrm{f}(a)}$	$\frac{i}{2}\Lambda_k^a + \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the anti–fundamental field	$\widetilde{m}_k^{\bar{\mathrm{f}}(a)}$	$\frac{i}{2}\Lambda_k^a - \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the adjoint field	$\widetilde{m}^{\mathrm{adj}(a)}$	$\frac{i}{2}C^{aa}$	diagonal element of the Cartan matrix
twisted mass of the bifundamental field	$\widetilde{m}^{\mathrm{b}(ab)}$	$\frac{i}{2}C^{ab}$	non–diagonal element of the Cartan matrix
FI–term for $U(1)$–factor of gauge group $U(N_a)$	τ_a	ϑ^a	boundary twist parameter for particle species a

Table 5.1: Dictionary in the Gauge/Bethe correspondence.

Literature. The gauge theory prerequisites are explained in sections 12.1, 12.2, 15.2 and 15.5 of the book *Mirror Symmetry* by K. Hori et al [1].

The 2d gauge/Bethe correspondence was introduced in [2, 3]. The dictionary is explained in detail in [4, 5].

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